A DISCUSSION ON THE THREE-DIMENSIONAL BOUNDARY VALUE PROBLEM FOR ELECTROMAGNETIC FIELDS

David Rankin
Institute of Earth and Planetary Physics
Dept. of Physics
University of Alberta
Edmonton , Alberta

Three-dimensional boundary value problems are difficult to solve. Indeed, while the separation of the scalar wave equation can be effected in 11 different coordinate systems, an analytic solution requires that the boundaries, both external and internal, possess the same symmetry as the coordinate system. Numerical methods are thus of great importance for the solution of such problems; however, despite the availability of high-speed, large memory digital computers, the solution to a significant three-dimensional problem is by no means trivial. It is unfortunate that the results reported here by Jones are invalid.

Jones discusses a model in which a three-dimensional island lies off a linear coastline where all the interfaces lie in the coordinate planes in a Cartesian system. A downward plane em wave polarized with the electric field parallel to the linear coastline, and two of the islands coast, is incident downward. This, of course, requires that the incident electric field is perpendicular to the other faces of the idealized Cartesian island.

The significant equation used by Jones for his finite difference calculations is

$$\frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} - \frac{\partial}{\partial x}\left[\frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right] = i\eta^2 E_x$$

where $\eta^2 = \sigma\mu\omega$. σ is the conductivity, $\mu = \mu_0$ is the permeability of the appropriate medium, and ω is the angular frequency of a Fourier component of the incident field. We see that the displacement current term is omitted, which is an excellent approximation for the parameters of interest in this model. In the similar equation for the y-component of the diffusion equation, the same considerations which we are about to discuss are equally valid.

It can be seen from the equation that the field components are related and thus any errors introduced will be propagated into all three components. The difficulty arises when E_x or E_y is perpendicular to one or other of the surfaces of discontinuity in σ. The continuity of current at the air-earth interface (upper surface), e.g., j_y, requires that E_y is discontinuous; i.e., $j_y = \sigma_1 E_{y1} = \sigma_2 E_{y2}$. Jones uses the average value of E_y at the interface in the belief that continuity of E_y is necessary in order to calculate $\partial E_y / \partial y$ in the first equation above. The calculations are made in the finite difference approximation. The result is that all the field components and the derivatives are grossly distorted in the region about the boundary and this distortion will be propagated outward into the surrounding regions. Since the magnetic field components are deduced from the electric through Maxwell's equations, they will also suffer a distortion. Self-consistency required that the correct electric field components are themselves deducible from the magnetic, clearly impossible in this case.

Furthermore, it is also necessary that if E_y is forced to be continuous, then j_y must be discontinuous. For the values used by Jones $j_{y1}/j_{y2} = \sigma_1 / \sigma_2 = 4 \times 10^3$. This would be in clear violation of the continuity condition on j which is used explicitly by Jones in his work and would in turn require that

$$\nabla \cdot \vec{j} = -\partial \rho / \partial t = j_1 - j_2 = -\nabla \cdot \partial \vec{D} / \partial t$$

which requires a by no means negligible displacement current.

The anti-skin effect and the nonzero values of E_z which Jones obtains at the air-earth interface may be due to a computer artifact as well as the incorrect computational procedures. In any event, the incorrect procedures could have been avoided since good numerical approximations can be obtained by taking one-sided derivatives of the discontinuous functions near the interfaces. The averaging of these derivatives would, of course, be a permissible procedure.

For a complete discussion of the nature of the discontinuity in the electric field in the two-dimensional case, see D'Erceville and Kunetz (1962). The relevance to the three-dimensional case is quite clear. The subject matter discussed here by Jones appears in the publications Jones and Pascoe (1972), Lines and Jones (1973a, 1973b), and Lines (1972).

References

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